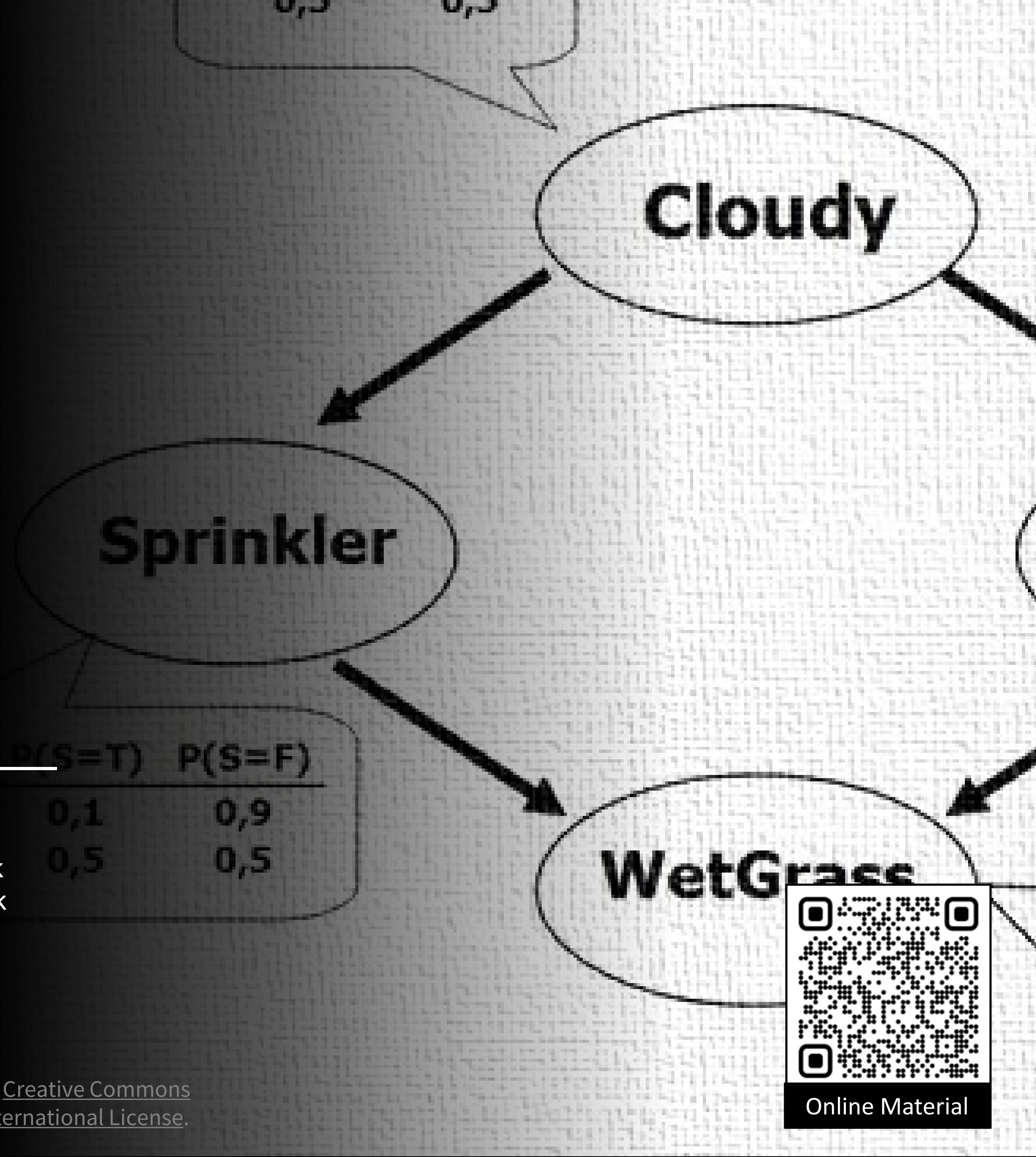
CS 5/7320
Artificial Intelligence

Probabilistic Reasoning
AIMA Chapter 13

Slides by Michael Hahsler
based on slides by Svetlana Lazepnik
with figures from the AIMA textbook



Online Material

Probability Theory Recap

- Notation: Prob. of an event $P(X = x) = P(x)$
  Prob. distribution $\boldsymbol{P}(X) = \langle P(X = x_1), P(X = x_2), \dots, P(X = x_n) \rangle$

- Product rule $P(x, y) = P(x|y)P(y)$
- Chain rule $\boldsymbol{P}(\mathrm{X_1}, \mathrm{X_2}, \dots, \mathrm{X_n}) = \mathbf{P}(\mathrm{X_1})\mathbf{P}(\mathrm{X_2}|\mathrm{X_1})\mathbf{P}(\mathrm{X_3}|\mathrm{X_1}, \mathrm{X_2}) \dots$
  $= \prod_{i=1}^{n} \boldsymbol{P}(X\_i|X_1, \dots, X_{i-1})$
- Conditional probability $P(x|y) = \frac{P(x,y)}{P(y)} = \alpha P(x, y)$

- Independence
  - $X \perp\!\!\!\perp Y$: X, Y are independent (written as $X \perp\!\!\!\perp Y$) if and only if:
    $\forall x, y: P(x, y) = P(x)P(y)$

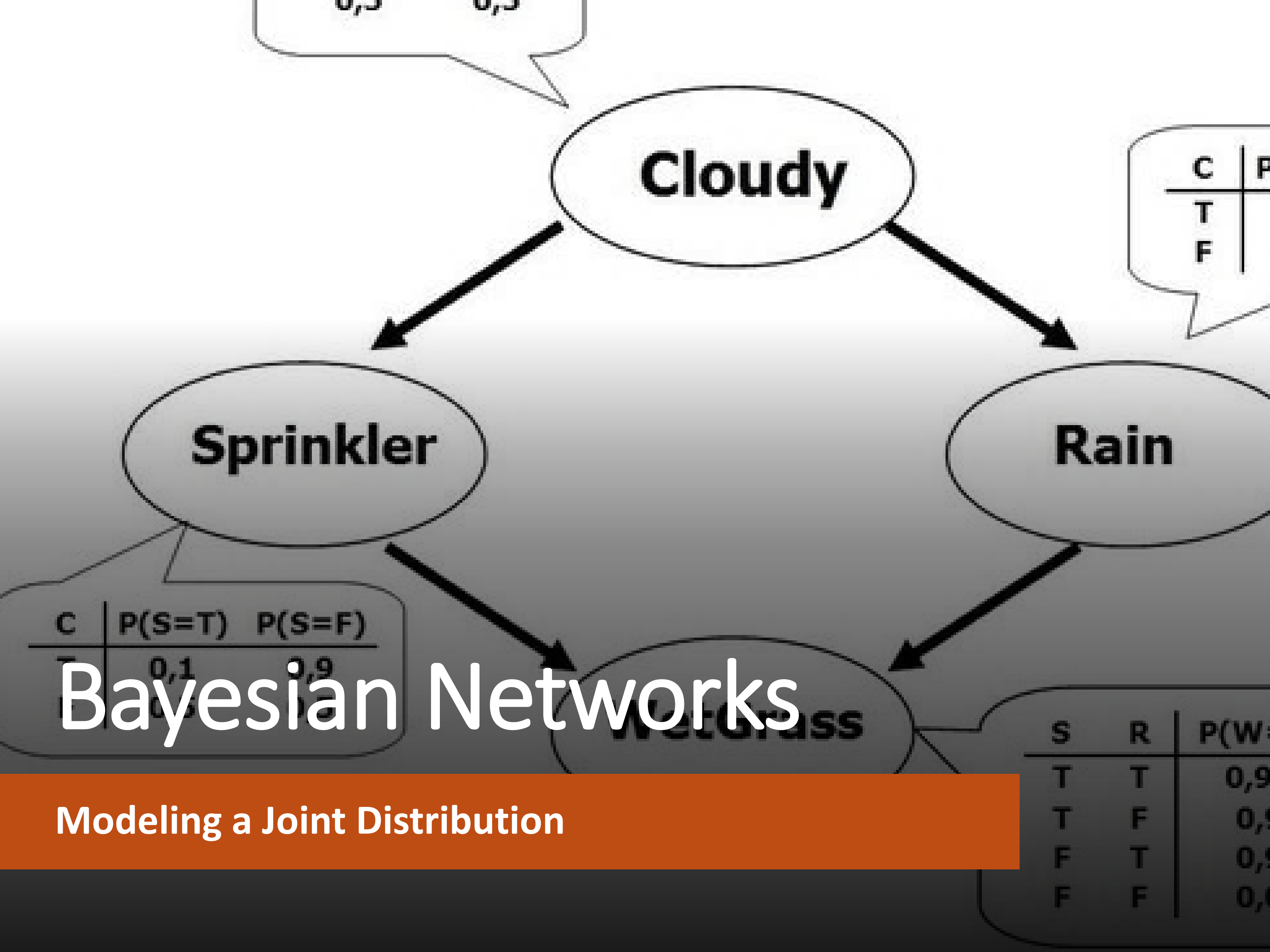
  - $X \perp\!\!\!\perp Y|Z$: X and Y are conditionally independent given Z if and only if:
    $\forall x, y, z: P(x, y|z) = P(x|z)P(y|z)$

Contents

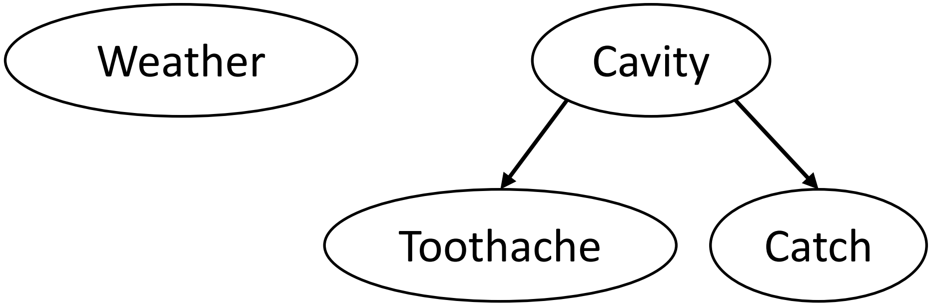
- Bayesian Networks to Specify Dependence
- Exact Inference
- Approximate Inference

Bayesian Networks

Modeling a Joint Distribution

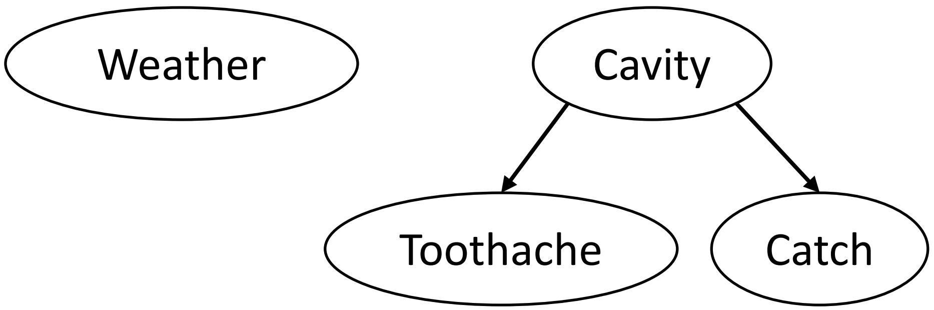


Bayesian Networks
(aka Belief Networks)



- A type of graphical model.
- A way to specify dependence between random variables.
- A compact specification of a full joint probability distribution.
- A general and important model to reason with uncertainty in AI.

Structure of Bayesian Networks



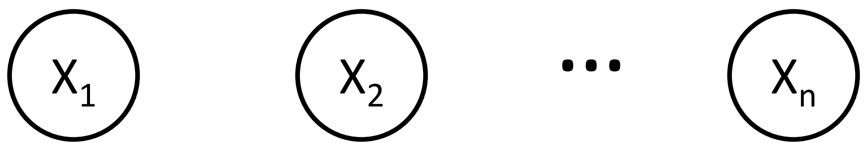
Nodes: Random variables
- Can be assigned (observed) or unassigned (unobserved)

Arcs: Dependencies
- An arrow from one variable to another indicates direct influence.
- Show independence
  - *Weather* is independent of the other variables (no connection).
  - *Toothache* and *Catch* are conditionally independent given *Cavity* (directed arc).
- Must form a directed *acyclic* graph (DAG)

A network with all random variables assigned represents a **state of the system**.

Example: N independent coin flips

Complete independence: no interactions between coin flips



$$P(X_1, X_2, \dots, X_n) = P(X_1)P(X_2)\dots P(X_n)$$

Joint probability distribution

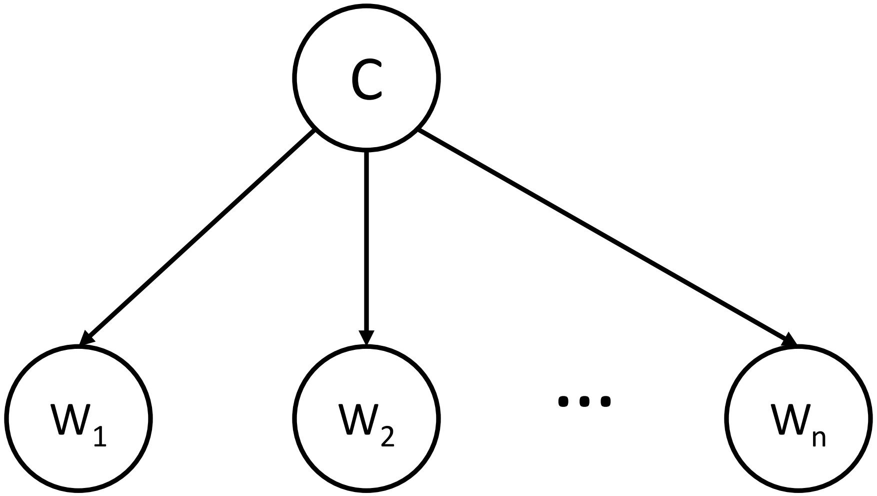
Marginal probability distributions

Example: Naïve Bayes spam filter

Random variables:

- C: message class (spam or not spam)
- $W_1, \ldots, W_n$: presence or absence of words comprising the message

Words depend on the class, but they are modeled conditional independent of each other given the class (= no direct connection between words).



$$P(W_1, W_2, \ldots, W_n|C) = P(W_1|C)P(W_2|C) \ldots P(W_n|C)$$

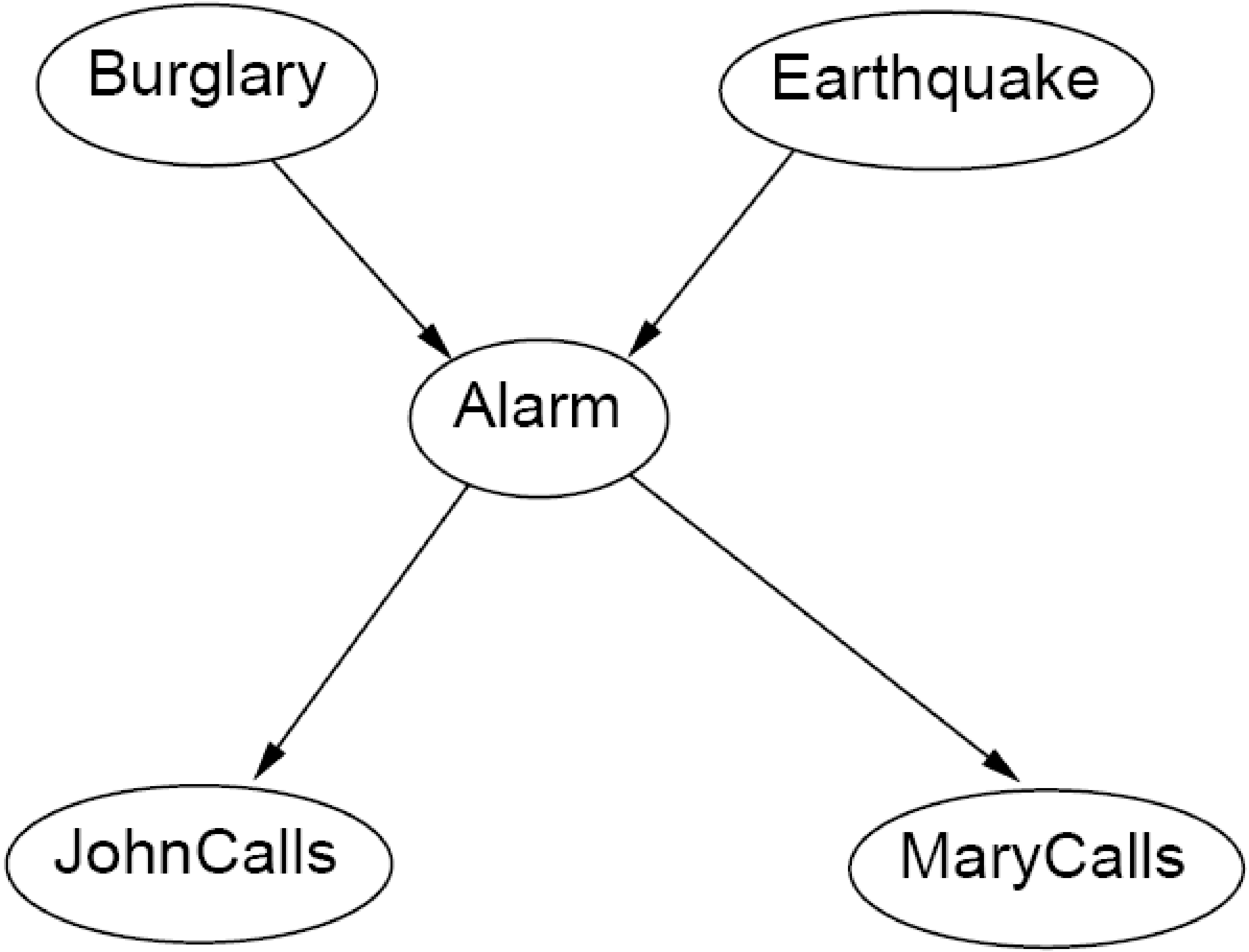
Example: Burglar Alarm

- **Description**: I have a burglar alarm that is sometimes set off by minor earthquakes. My two neighbors, John and Mary, promised to call me at work if they hear the alarm
- Example inference task: suppose Mary calls and John doesn't call. What is the probability of a burglary?

- What are the random variables?
  - Burglary, Earthquake, Alarm, JohnCalls, MaryCalls

- What are the direct influence relationships?
  - A burglar can set off the alarm
  - An earthquake can set off the alarm
  - The alarm can cause Mary to call
  - The alarm can cause John to call

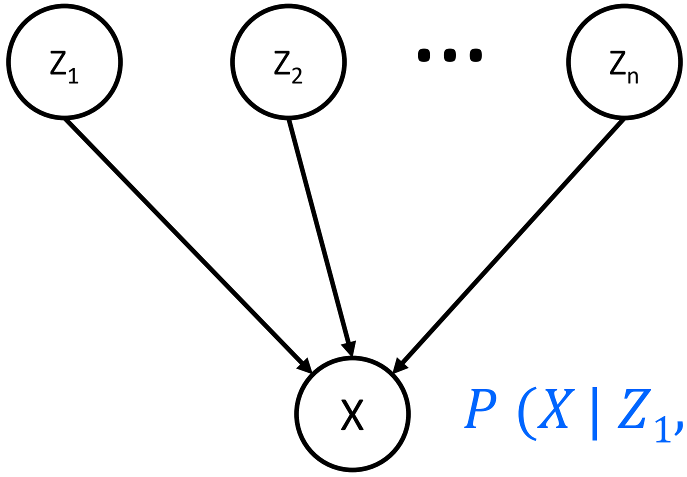
Example: Burglar Alarm as a Network



What are the model parameters?

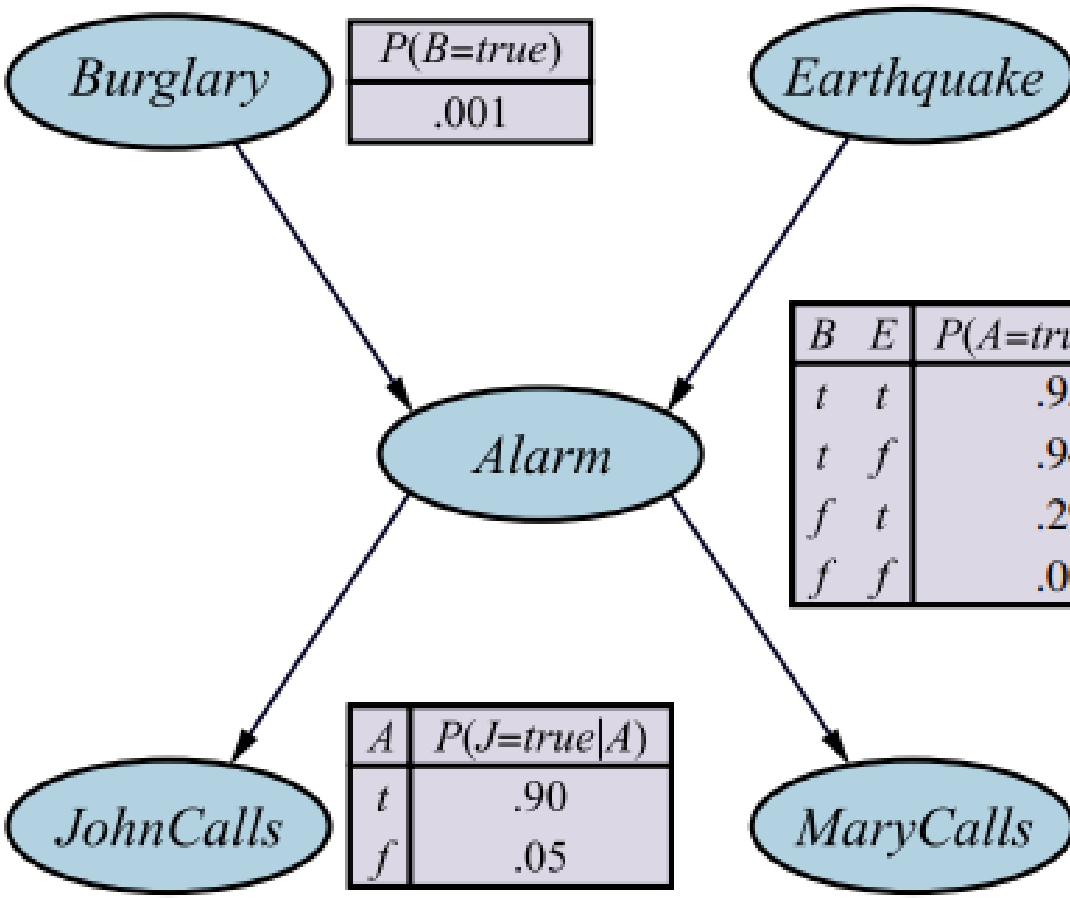
Parameters: Conditional probability tables

To specify the full joint distribution, we need to specify a *conditional* distribution for each node given its parents as a conditional probability table (CPT): $P\ (X\ |\ Parents(X))$



$P\ (X\ |\ Z_1, \ldots, Zn)$

Example: Burglar Alarm with CPTs



No parents

$P(B{=}true)$
.001

$P(E{=}true)$
.002

2 parents

B	E	$P(A{=}true \mid B,E)$
t	t	.95
t	f	.94
f	t	.29
f	f	.001

1 parent

A	$P(J{=}true \mid A)$
t	.90
f	.05

A	$P(M{=}true \mid A)$
t	.70
f	.01

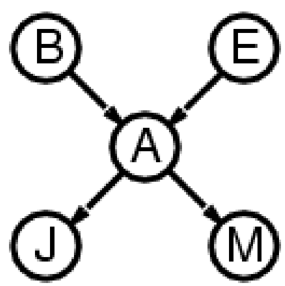
The joint probability distribution

- For each node X_i, we know $P(X_i \mid Parents(X_i))$
- How do we get the full joint distribution $P(X_1, \ldots, X_n)$?

- Using chain rule:

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | X_1, \ldots, X_{i-1}) = \prod_{i=1}^{n} P(X_i | Parents(X_i))$$

- Example:



Construct following arrows

$$P(J, M, A, B, E) = P(B)\, P(E)\, P(A \mid B, E)\, P(J \mid A)\, P(M \mid A)$$

Dependence

- Example: *causal chain*

X: Low pressure

Y: Rain

Z: Traffic

- Are X and Z independent?

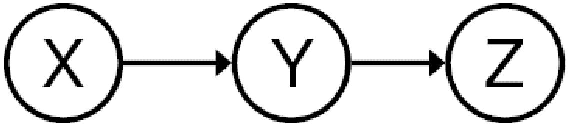
1. Conditioning: $P(X,Y,Z) = P(X)P(Y|X)P(Z|Y)$
2. Marginalize over y: $P(X,Z) = \sum_y P(X)P(y|X)P(Z|y)$

$$= P(X)\sum_y P(Z|y)P(y|X) \neq P(X)P(Z)$$

X and Z are **not** independent!

Conditional independence

- Example: *causal chain*



X: Low pressure

Y: Rain

Z: Traffic

- Is Z independent of X given Y?

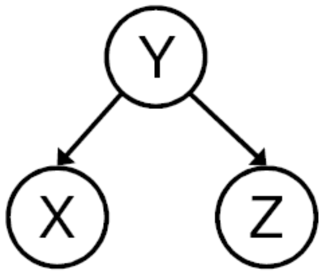
1. Conditioning: $P(X,Z|Y) = \frac{P(X,Y,Z)}{P(Y)} = \frac{P(X)P(Y|X)P(Z|Y)}{P(Y)}$
2. 2. Bayes' rule: $= \frac{P(X)\frac{P(X|Y)P(Y)}{P(X)}P(Z|Y)}{P(Y)} = P(X|Y)P(Z|Y)$

= Definition of conditional independence

X and Z are conditionally independent given Y

Conditional independence cont.

- Common cause



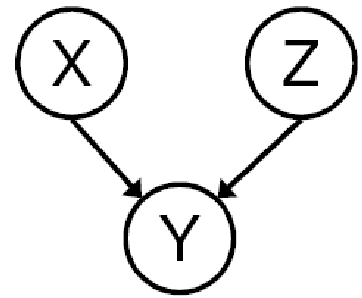
Y: Project due

X: Newsgroup busy

Z: Lab full

- Are X and Z independent?
  - No
- Are they conditionally independent given Y?
  - Yes

- Common effect



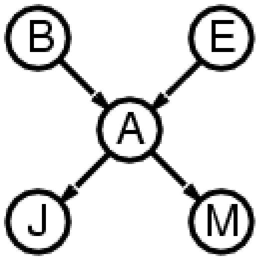
X: Raining

Z: Ballgame

Y: Traffic

- Are X and Z independent?
  - Yes
- Are they conditionally independent given Y?
  - No

Compactness



- Suppose we have a Boolean variable X_i with k Boolean parents. How many rows does its conditional probability table have?
  - 2^k rows for all the combinations of parent values, each row requires one number p for X_i = true
- If each variable has no more than k parents, how many numbers does the complete network require?
  - $O(n \cdot 2^k)$ numbers – vs. $O(2^n)$ for the full joint distribution
  - This reduces the complexity from exponential to linear in n!

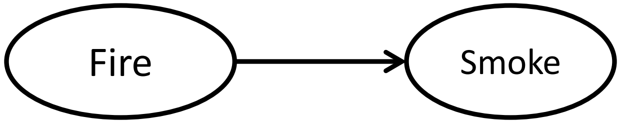
- Example: How many nodes for the burglary network?

  $1 + 1 + 4 + 2 + 2 = 10$ numbers
  (vs. specification of the complete joint probability $2^5 - 1 = 31$)

Constructing Bayesian networks

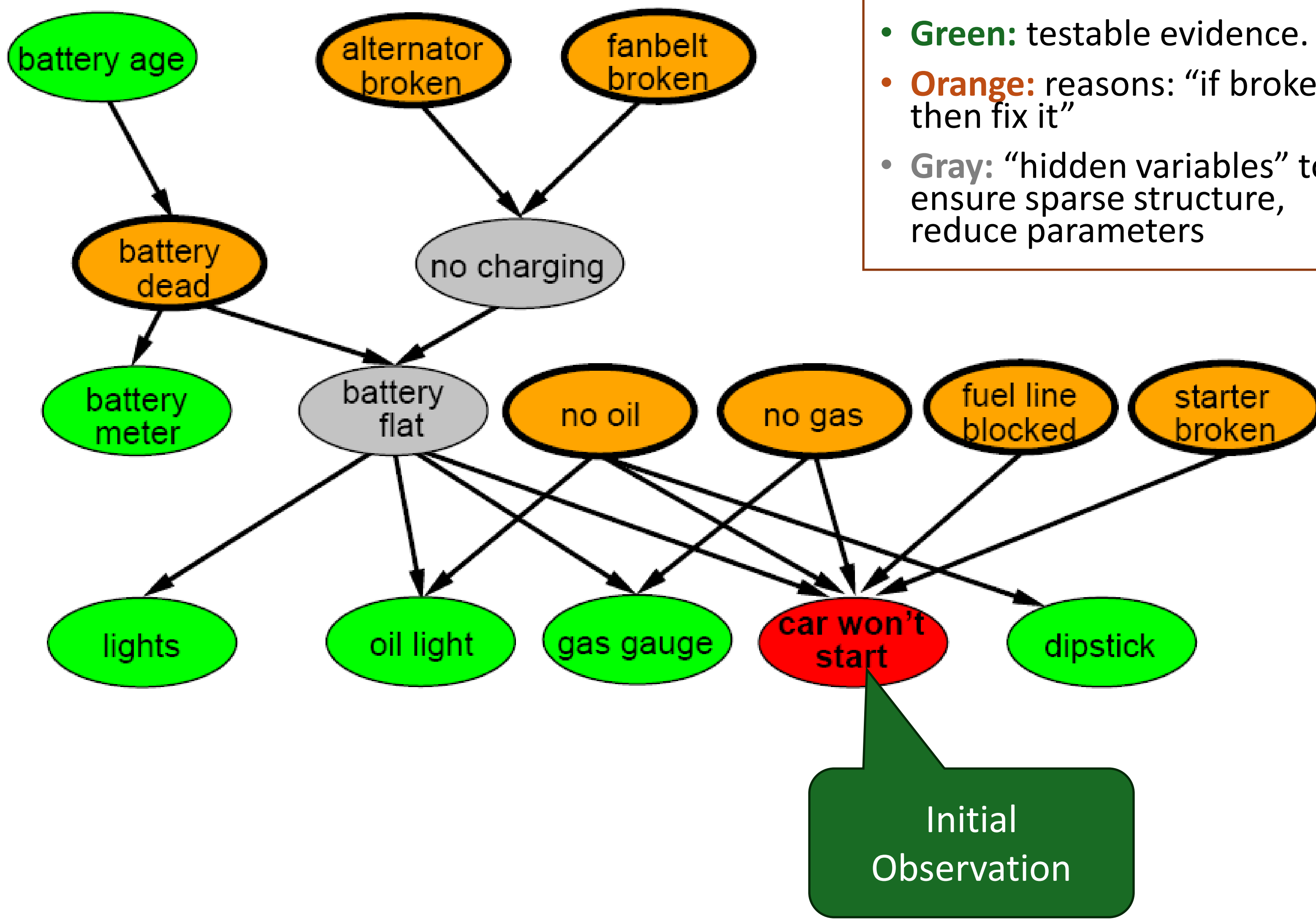
1. Choose an ordering of variables $X_1, \ldots, X_n$
2. For $i = 1$ to n
   - add X_i to the network
   - select parents from $X_1, \ldots, X_{i-1}$ such that
     $P(X_i \mid Parents(X_i)) = P(X_i \mid X_1, \ldots X_{i-1})$
     that is, add a connection only from nodes it directly depends on.

Note: There are many ways to order the variables. Networks are typically constructed by domain experts with causality in mind. E.g., Fire causes Smoke:



The resulting network is sparse and conditional probabilities are easier to judge because they represent causal relationships.

A more realistic Bayes Network: Car diagnosis



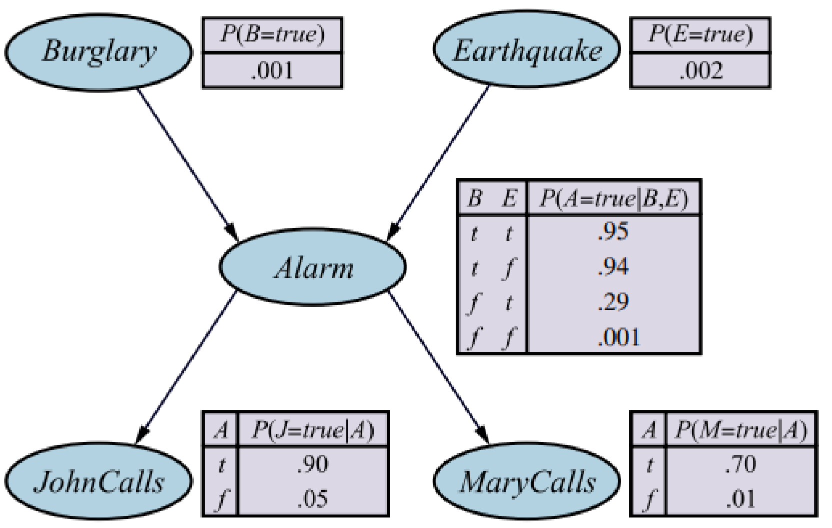
- **Initial observation**: car won't start.
- **Green:** testable evidence.
- **Orange:** reasons: "if broken, then fix it"
- **Gray:** "hidden variables" to ensure sparse structure, reduce parameters

Summary

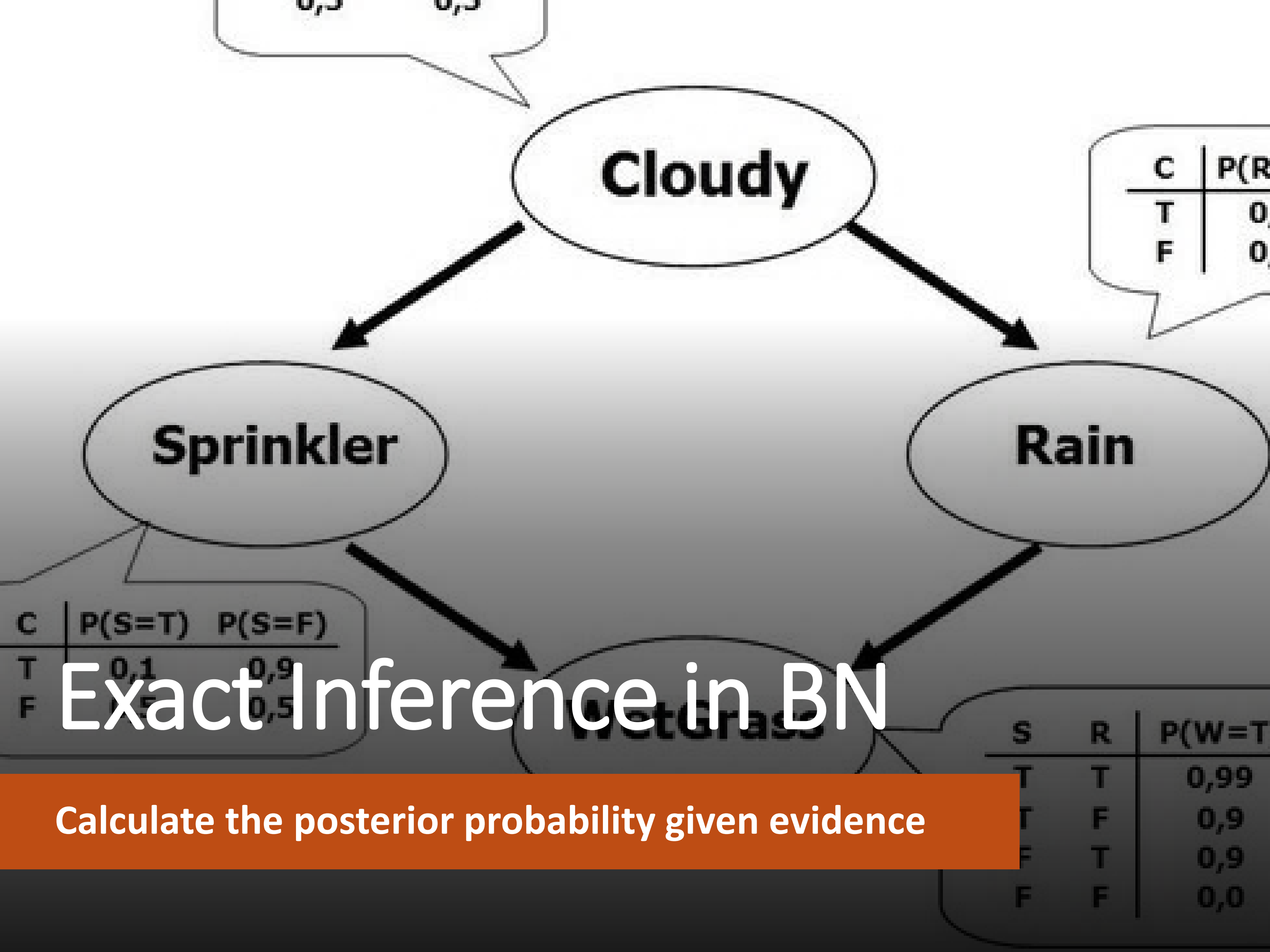
- Bayesian networks provide a natural representation for joint probabilities used to calculated conditional probabilities used in inference.
- Conditional independence (induced by causality) reduces the number of needed parameters.

$P(B, E, A, J, M)$ is defined by

- Representation
  - Topology
  - Conditional probability tables
  - Generally easy for domain experts to construct



Exact Inference in BN



Calculate the posterior probability given evidence

Exact Inference

Goal

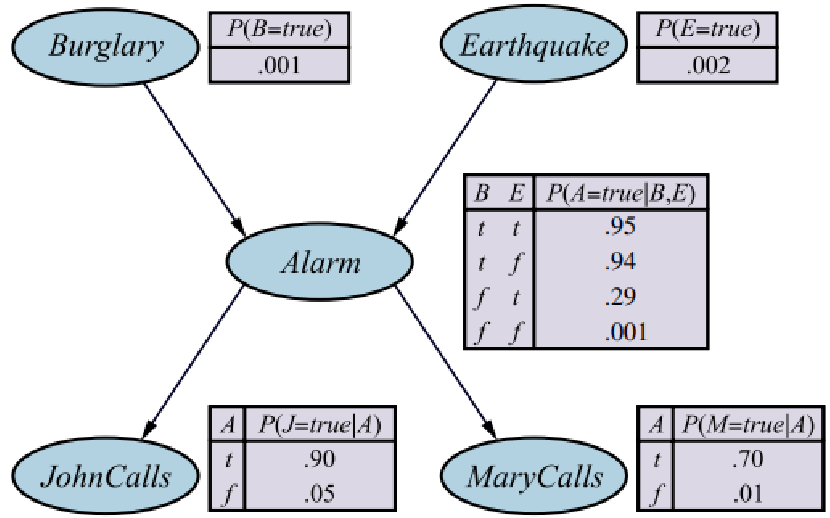
- Query *variables:* X
- *Evidence* (*observed*) variables: $E = e$
- *Set of unobserved* variables: Y
- Calculate the probability of X given e.

If we know the full joint distribution $P(X, E, Y)$, we can infer X by:

$$\boldsymbol{P}(X|E=e) = \frac{\boldsymbol{P}(X,e)}{P(e)} \propto \sum_{y} \boldsymbol{P}(X,e,y)$$

Sum over values of unobservable variables = marginalizing them out.

Exact inference: Example



Assume we can observe being called and the two variables have the values j and m. We want to know the probability of a burglary.
Query: $P(B \mid j, m)$ with unobservable variables: Earthquake, Alarm

$$P(b|j,m) = \frac{P(b,j,m)}{P(j,m)} \propto \sum_e \sum_a P(b,e,a,j,m)$$

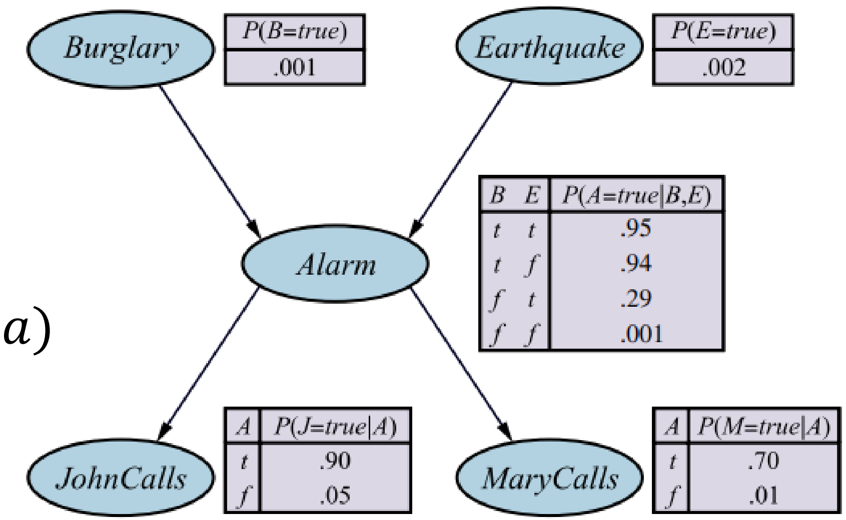
$$= \sum_e \sum_a P(b)P(e)P(a|b,e)P(j|a)P(m|a)$$

$$= P(b) \sum_e P(e) \sum_a P(a|b,e)\, P(j|a)P(m|a)$$

Full joint probability and marginalize over E and A

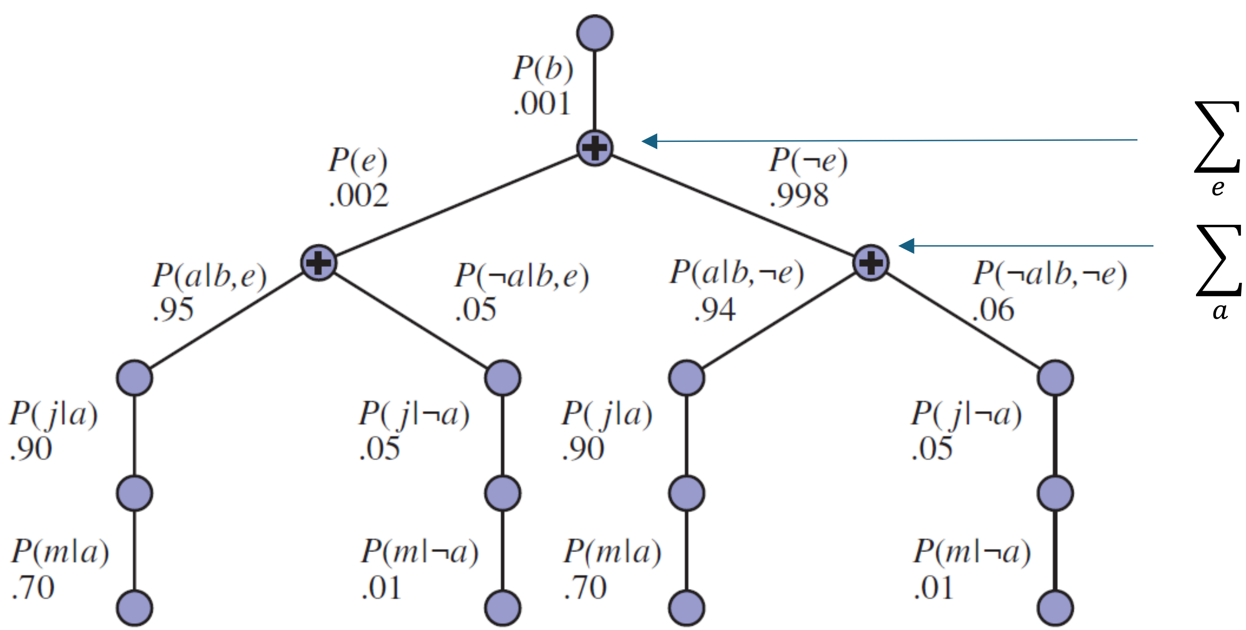
Exact inference: Example

$$P(b|j,m)$$
$$\propto P(b)\sum_{e} P(e) \sum_{a} P(a|b,e)\, P(j|a)P(m|a)$$



Evaluatior

(lines represe



Issues with Exact Inference in AI

$$\boldsymbol{P}(X|E=e) = \frac{\boldsymbol{P}(X,e)}{P(e)} \propto \boxed{\sum_{y}} \boxed{\boldsymbol{P}(X,e,y)}$$

Problems

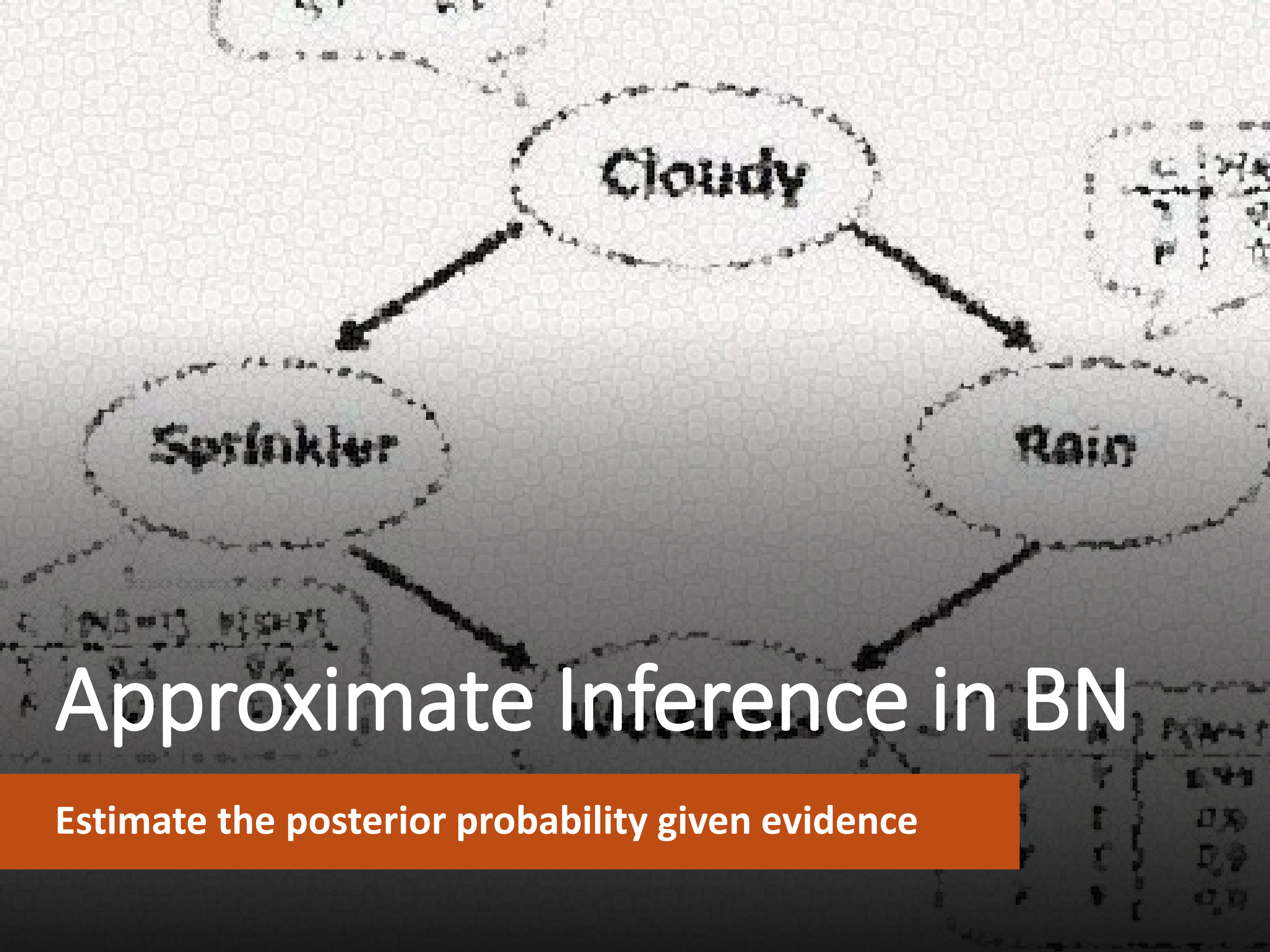
1. **Full joint distributions are too large** to store.

   Bayes nets provide significant savings for representing the conditional probability structure.

2. Marginalizing out many unobservable variables Y may involve **too many summation terms**.

   This summation is called **exact inference by enumeration**. Unfortunately, it does not scale well (#p-hard).

In praxis, **approximate inference by sampling** is used.



Approximate Inference in BN

Estimate the posterior probability given evidence

BN as a Generative Model

Bayesian networks can be used as ***generative models.***

Allows us to efficiently generate samples from the joint distribution.

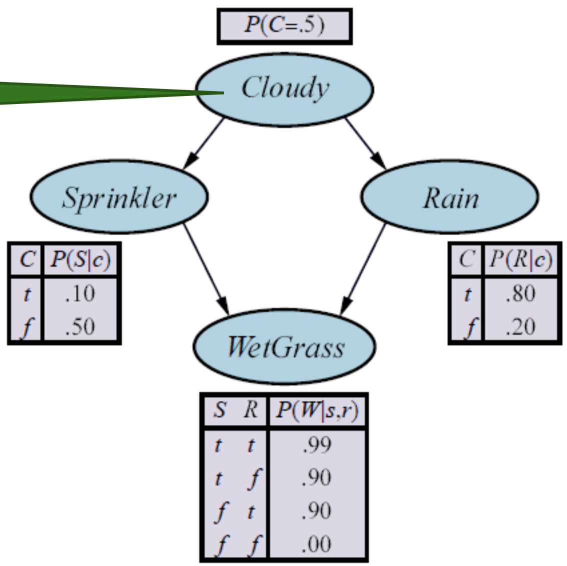
Idea: Generate samples from the network to estimate joint and conditional probability distributions.

Prior-Sample Algorithm to Create a Sample (Event)

function PRIOR-SAMPLE(bn) **returns** an event sampled from the prior specified by bn
  inputs: bn, a Bayesian network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

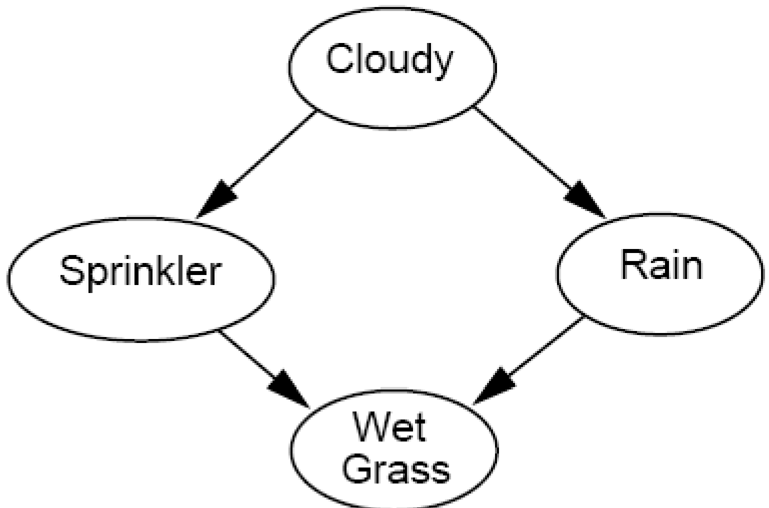
  $\mathbf{x} \leftarrow$ an event with n elements
  for each variable X_i **in** $X_1, \ldots, X_n$ **do**
    $\mathbf{x}[i] \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
  return $\mathbf{x}$

We need to start with the random variables that have no parents.



Example: Sampling from a Bayesian Network

P(C)
.50

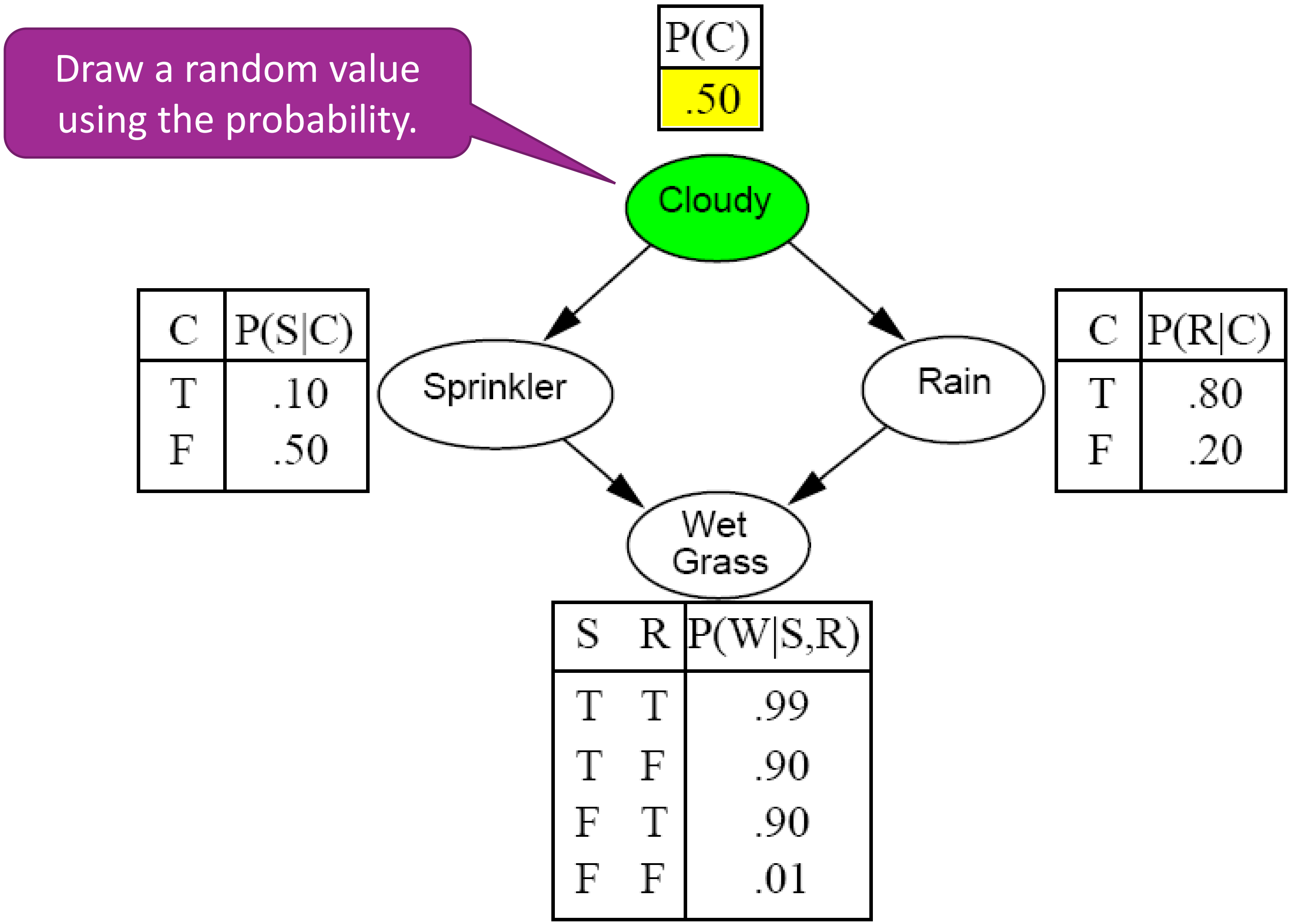


C	P(S|C)
T	.10
F	.50

C	P(R|C)
T	.80
F	.20

S	R	P(W|S,R)
T	T	.99
T	F	.90
F	T	.90
F	F	.01

Example: Sampling from a Bayesian Network



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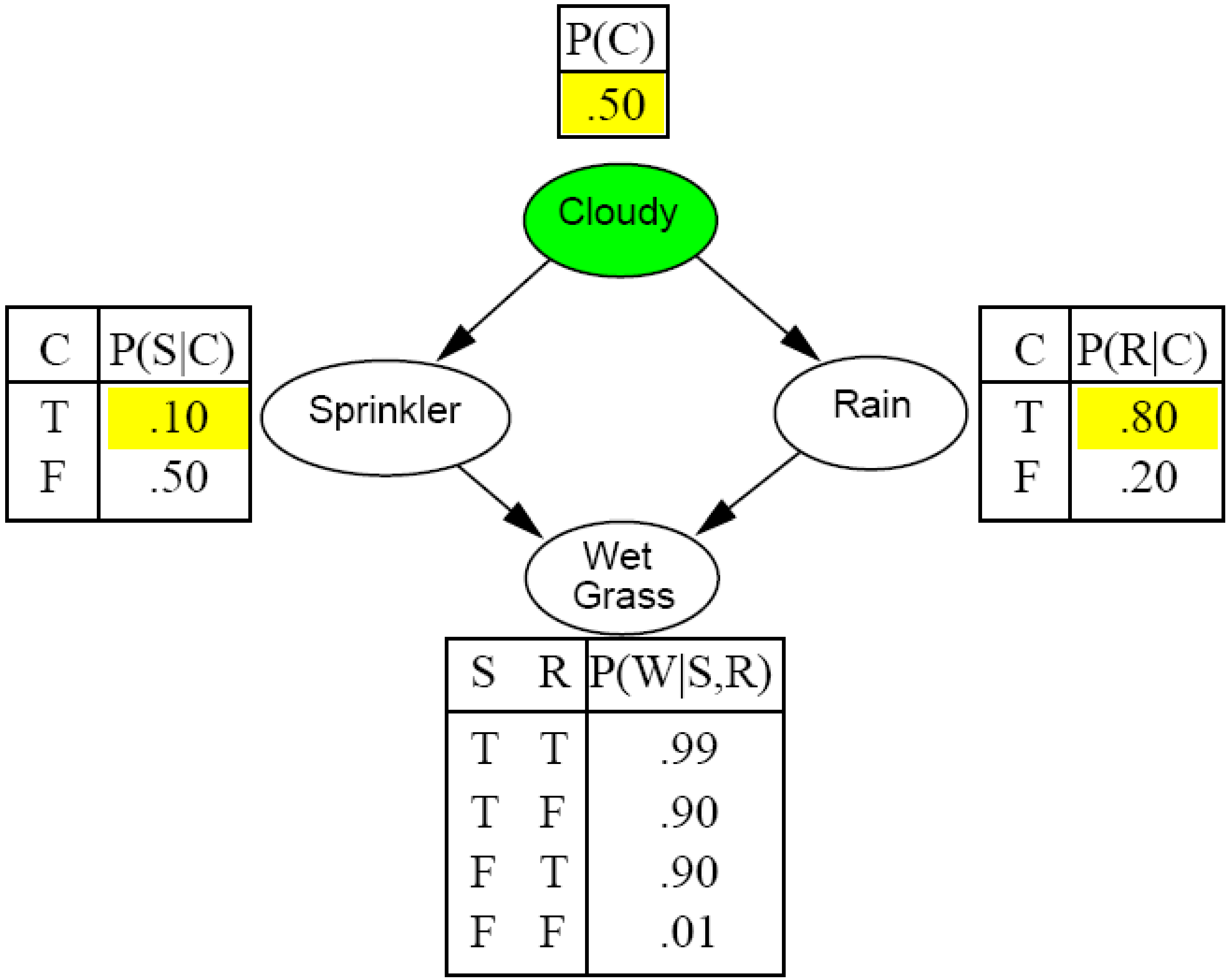

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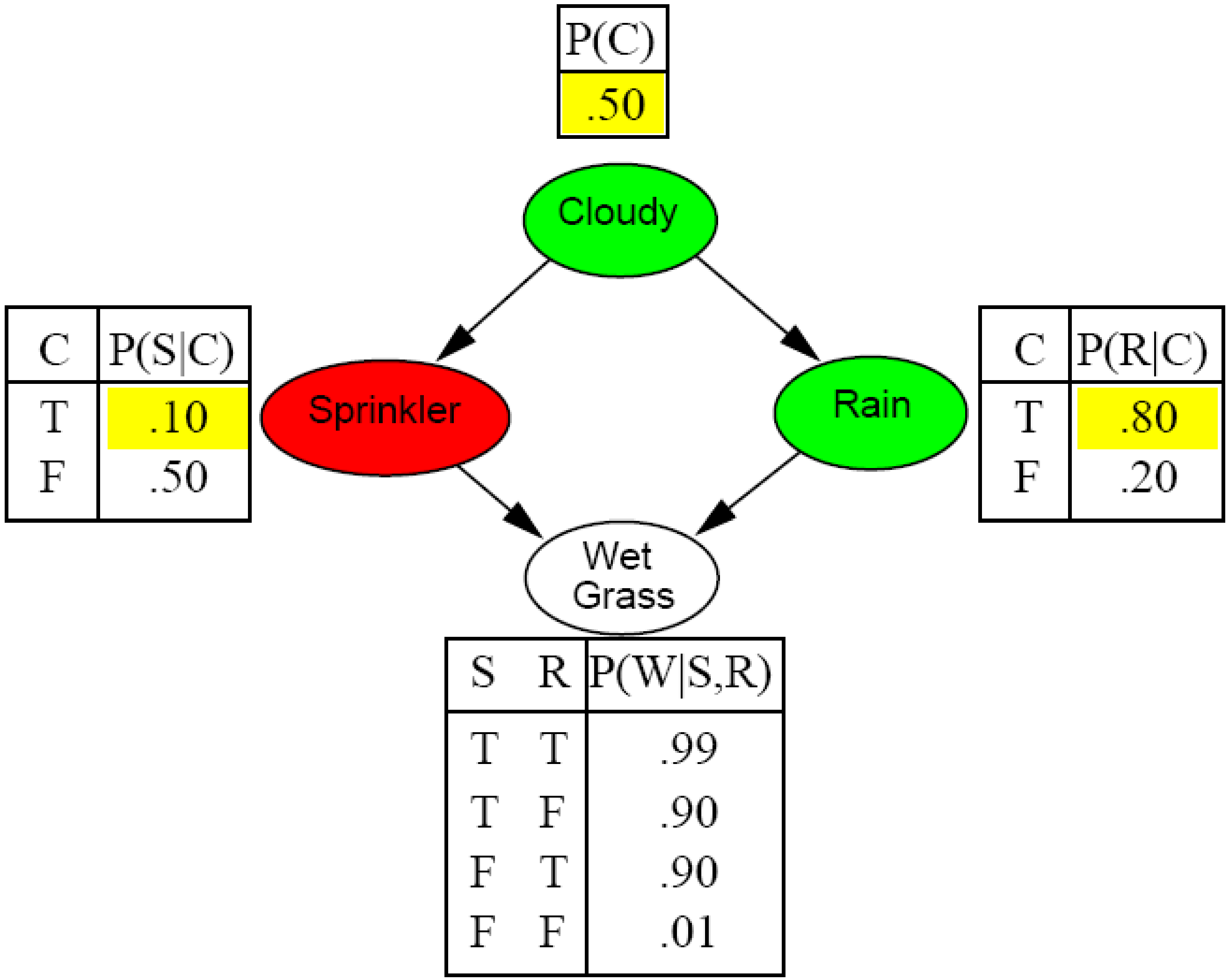

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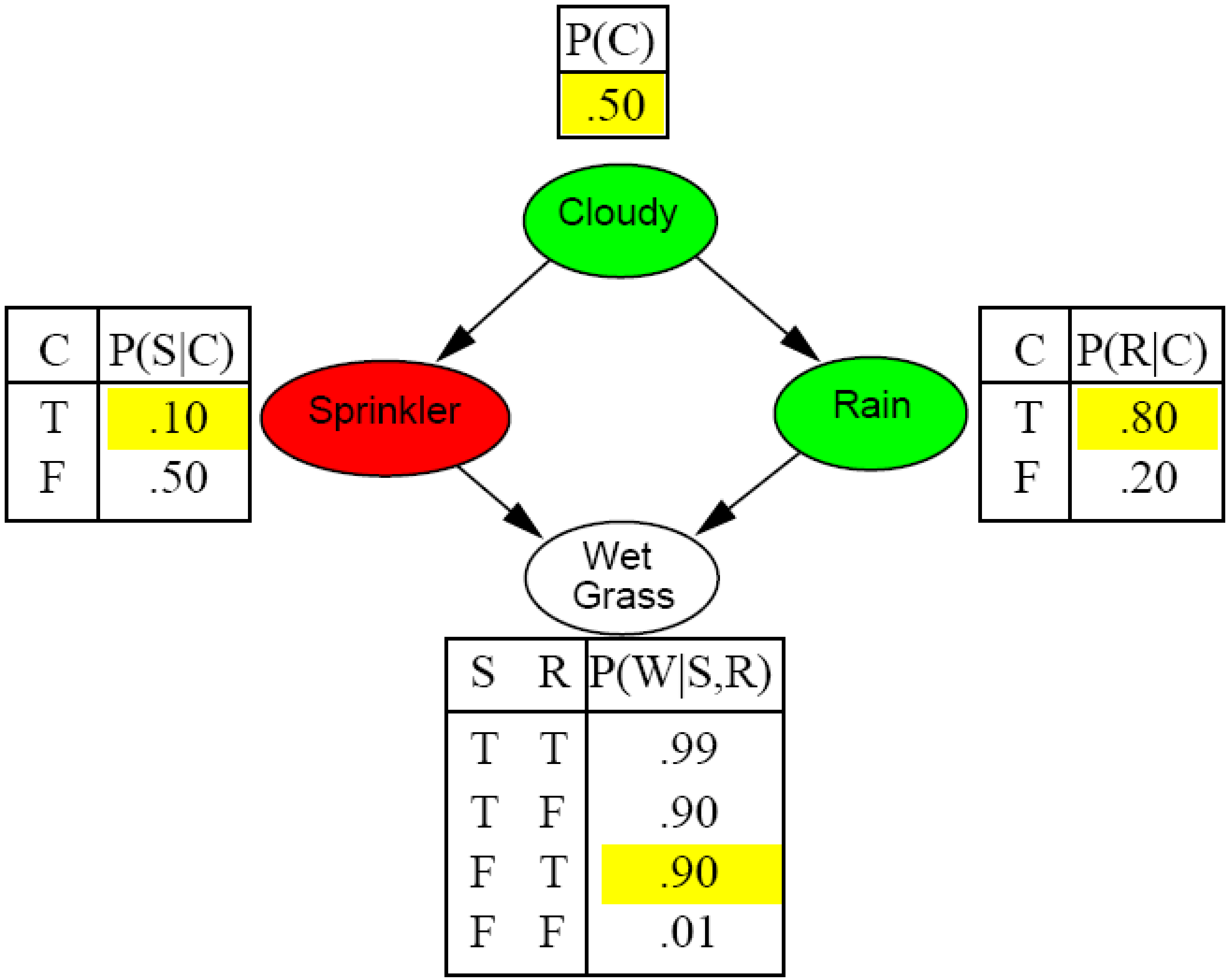

Example: Sampling from a Bayesian Network

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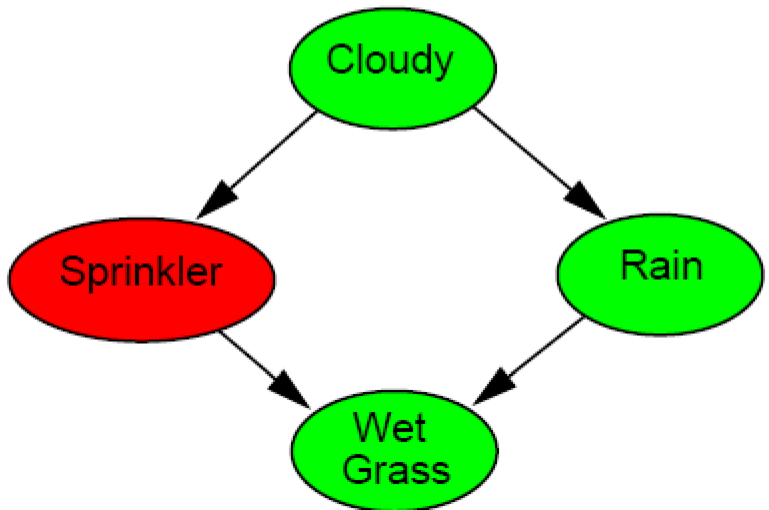


C	P(S|C)
T	.10
F	.50

C	P(R|C)
T	.80
F	.20

S	R	P(W|S,R)
T	T	.99
T	F	.90
F	T	.90
F	F	.01

Prior Sample returns the event:

[C = True, S = False, R = True, W = True]

Estimating the Joint Probability Distribution

Sample *N* times and determine $N_{PS}(x_1, x_2, \ldots, x_n)$, the count of how many times Prior-Sample produces event $(x_1, x_2, \ldots, x_n)$.

$$\hat{P}(x_1, x_2, \ldots, x_n) = \frac{N_{PS}(x_1, x_2, \ldots, x_n)}{N}$$

The marginal probability of partially specified event (some x values are known) can also be calculates. E.g.,

$$\hat{P}(x_1) = \frac{N_{PS}(x_1)}{N}$$

Estimating Conditional Probabilities: **Rejection sampling**

Sample *N* times and **ignore the samples that are not consistent with the evidence e.**

$$\hat{P}(X|e) = \alpha N_{PS}(X, e) = \frac{N_{PS}(X, e)}{N_{PS}(e)}$$

Issue: What if e is a rare event?

- Example: burglary $\wedge$ earthquake
- Rejection sampling ends up throwing away most of the samples. This is very inefficient!

Estimating Conditional Probabilities: **Rejection sampling**

function REJECTION-SAMPLING(X, **e**, bn, N) **returns** an estimate of $\mathbf{P}(X \mid \mathbf{e})$
  inputs: X, the query variable
    e, observed values for variables **E**
    bn, a Bayesian network
    N, the total number of samples to be generated
  local variables: **C**, a vector of counts for each value of X, initially zero

  for j = 1 **to** N **do**
    x ← PRIOR-SAMPLE(bn)
    if x is consistent with **e then**
      C[j] ← **C**[j]+1 where x_j is the value of X in **x**
  return NORMALIZE(**C**)

We throw away many samples if e is rare!

Estimating Conditional Probabilities: **Importance sampling** (likelihood weighting)

Goal: Avoid the need of rejection sampling to throw out samples.

1. Fix the evidence $\boldsymbol{E = e}$ for sampling and estimate the probably for the non-evidence variables using prior-sampling.

$$Q_{WS}(x)$$

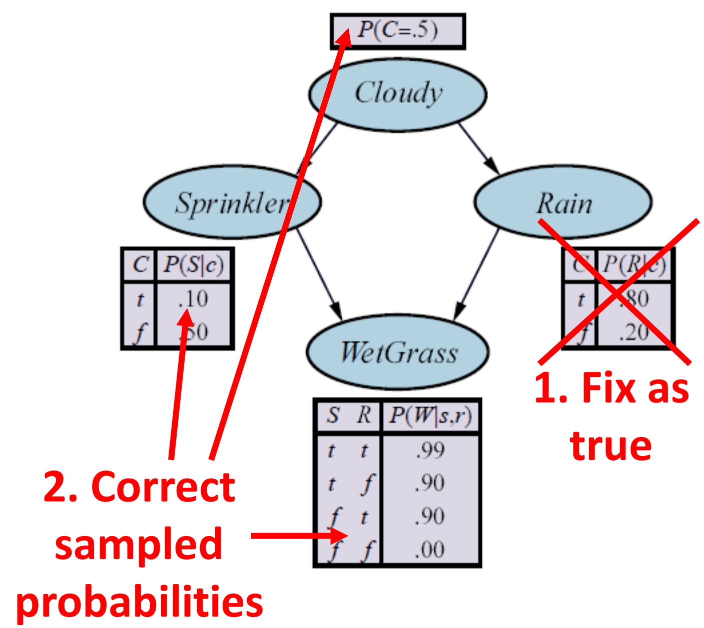
2. Correct the probabilities using weights

$$P(x|e) = w(x)Q_{WS}(x)$$

Turns out the weights in this case can be easily calculated

$$w(x) = \frac{1}{P(e)} \prod_{i=1}^{m} P(e_i|parents(E_i))$$

Example: Evidence = it rains



Estimating Conditional Probabilities: Markov Chain Monte Carlo Sampling (MCMC)

- **Generates a sequence of samples** instead of creating each sample individually from scratch.
- Create a state by making random changes to the current state. The sequence of states forms a random process called a **Markov Chain** (MC).
- The MCs stationary distribution turns out to be the posterior distribution of the non-evidence variables.
- Estimate the stationary distribution using **Monte Carlo** simulation by counting how often each state is reached and normalize to obtain probability estimates.
- Algorithms:
  1. Gibbs sampling (works well for BNs)
  2. Metropolis-Hastings sampling

Note: Simulated annealing belongs to the family of MCMC algorithms.

Gibbs sampling in Bayes Networks

```
function GIBBS-ASK(X, e, bn, N) returns an estimate of P(X | e)
  local variables: C, a vector of counts for each value of X, initially zero
                   Z, the nonevidence variables in bn
                   x, the current state of the network, initialized from e

  initialize x with random values for the variables in Z
  for k = 1 to N do
     choose any variable Z_i from Z according to any distribution ρ(i)
     set the value of Z_i in x by sampling from P(Z_i | mb(Z_i))
     C[j] ← C[j] + 1 where x_j is the value of X in x
  return NORMALIZE(C)
```

Random State

Change one variable

Count

- $mb(Z_i)$ is the Markov blanket of random variable Z_i (all variables it can be dependent of, i.e., parents, children and parents of children).

$$P(z_i|mb(Z_i)) = \alpha P(z_i|parents(Z_i)) \prod_{Y_i \in children(X_i)} P(y_i|parents(Y_j))$$

Gibbs Sampling: Example

Find
$P(Rain \mid Sprinkler = true, WetGrass = true)$.

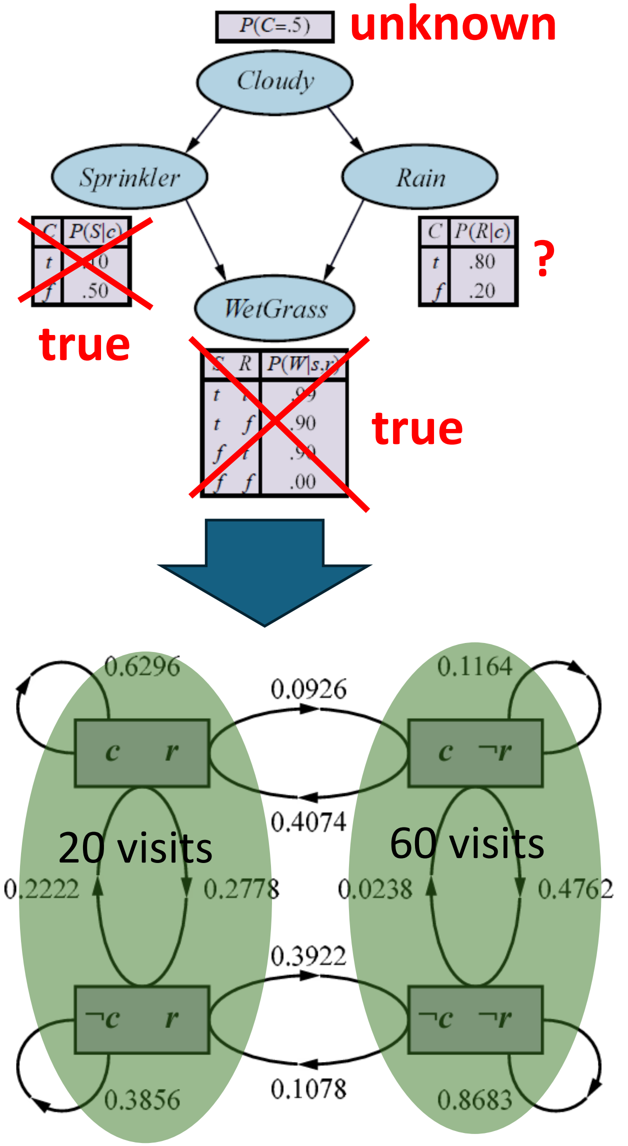
Determine states and calculate transition probabilities of the Markov chain for changing one variable using $P(z_i \mid mb(Z_i))$.

The algorithm randomly wanders around in this graph using the stated transition probabilities.

Assume that we observe 20 states with $Rain = true$ and 60 with $rain = false$:

$$NORMALIZE(\langle 20{,}60\rangle) = \langle 0.25{,}0.75\rangle$$

$P(Rain \mid Sprinkler = true, WetGrass = true) \approx 0.75$



Note the self-loops: the state stays the same when either variable is chosen and then resamples the same value it already has.

Conclusion

- Bayesian networks provide an efficient way to store a complete probabilistic model by exploiting (conditional) independence between variables.
- Inference means querying the model for a conditional probability given some evidence.
- Exact inference is difficult, for all but tiny models.
- State of the art is to use approximate inference by sampling from the model.
- Software libraries provide general inference engines.